Arithmetic: The Last Holdout

Ms. Burns urges mathematics teachers to stop teaching standard algorithms and to start having children invent their own methods. She also explores, along the way, some of the ramifications of this major change in arithmetic instruction.

BY MARILYN BURNS

THE CALL for reforming mathematics teaching has been made loudly and strongly. In 1989 two important documents appeared – *Curriculum and Evaluation Standards for School Mathematics*, published by the National Council of Teachers of Mathematics (NCTM), and *Everybody Counts: A Report to the Nation on the Future of Mathematics Education*, sponsored by the National Research Council and published by National Academy Press. In 1990 the Mathematical Sciences Education Board released *Reshaping School Mathematics*. Since then, other publications have followed, and articles have appeared in many education journals. The overall message has been a consistent one: teach the children to solve problems, reason, communicate, value mathematics, and become confident in their ability to do mathematics. Teaching for understanding is in; learning rote skills is out.

MARILYN BURNS is president of Marilyn Burns Education Associates, Sausalito, Calif.

Illustration by Les Kanturek

The call for change has also received attention outside the education community, via reports in the general media. The November 1989 issue of *Parenting* magazine included an article, "Math Comes Alive," informing parents that they should expect their children to experience math teaching that extends beyond "using their time simply to memorize arithmetic procedures and multiplication tables." In the fall of 1990, *Newsweek* published a special issue on education that included "Creating Problems," an article devoted to math teaching. The article stated: "It's time to minimize rote learning and concentrate on teaching children how to think." The *Wall Street Journal* issue of 11 September 1992 included an article titled "Reinventing Math." It was subtitled: "Active learning promises radical changes, as teachers say the rote approach doesn't add up." The article reported that "math education is undergoing change that promises to be far more pervasive and persistent than past reform movements."

THE SITUATION IN THE SCHOOLS

To some teachers, the need for change is crystal clear and the call for it, long overdue. For many teachers, however, the need for reform isn't at all compelling. A survey conducted by Horizon Research of Chapel Hill, North Carolina, reported that less than one-third of elementary teachers and only about one-half of secondary teachers surveyed said that they were "well aware" of the new math standards. In "Mathematical Power to the People," an essay in the August 1990 issue of the *Harvard Educational Review*, Alan Bishop wrote that, while the goals of the new teaching recommendations were "undoubtedly worth striving for," he felt that the reform movement was "high on leadership but low on followership."

For many teachers, the textbooks and standardized tests give the "real" message about what *should* be taught in the classroom. Judging from the content of textbook lessons and standardized test items, the message is that *children must develop proficiency with paper-and-pencil arithmetic calculations*. To respond to this goal, teachers have traditionally taught arithmetic skills out of context, stressed the memorization of rules, and relied on worksheet drill on paper-and-pencil algorithms. Yet the NCTM standards put these very methods on their hit list of instructional practices that should receive *less* attention under the new guidelines.

> Is the call for change just another of education's many bandwagons that we must acknowledge superficially until it passes by?

Teachers remain skeptical. Is the call for change just another of education's many bandwagons that we must acknowledge superficially until it passes by and the dust settles? If not, how do the new recommendations mesh with the demands of current textbooks and standardized tests?

A NEW LOOK

I am a teacher who has embraced the call for change fully and completely. I've made shifts in my teaching so that helping children learn to think, reason, and solve problems has become the primary objective of my math instruction. I do not give assignments on which the children can be successful by performing rote operations whether or not they understand what they're doing. I do not give timed tests on basic facts. I make calculators available for students to use at all times. I incorporate a variety of manipulative materials into my instruction.

I do not rely on textbooks because textbooks, for the most part, encourage "doing the page" rather than "doing mathematics." I avoid dittos and workbook pages because I want children to organize their ideas on paper for themselves and to use symbolic representations in ways that mean something to them. I don't want children to think that filling in boxes suffices for doing mathematics. I stress communication in math class by encouraging my students to talk and write. I have students work in pairs, in small groups, and individually, and I lead whole-class discussions regularly. I plan instructional units that provide students with choices so that they have an investment in and control over their math learning.

Each of these changes was a big one for me. Each required me to give up a time-honored belief, to deepen my understanding of mathematics and of how children learn, and to strengthen my teaching skills. The changes didn't occur all at once; they evolved.

MAKING BIG CHANGES

But the most dramatic change that I've made in my career is in the way I teach arithmetic. It was the last holdout. As a beginning teacher 30 years ago, I was comfortable having my students chant, "Divide, multiply, subtract, bring down," when learning long division. However, as I became more committed to the notion that students should do only what makes sense to them, I could no longer continue to have children do things they didn't understand. I gave up the chant.

But I remained perplexed. How could middle school students be helped to understand why inverting and multiplying is a logical procedure for dividing fractions? How could second-graders, whose understanding of the place-value structure of the number system is typically fragile, learn to understand why borrowing makes sense for subtraction? How could fourth-graders be taught the logic of the algorithm for long division?

I came to several conclusions. Imposing the standard arithmetic algorithms on children is pedagogically risky. It interferes with their learning, and it can give students the idea that mathematics is a collection of mysterious and often magical rules and procedures that must be memorized and practiced. Teaching children sequences of prescribed steps for computing focuses their attention on following the steps, rather than on making sense of numerical situations. It gives students the message that getting correct answers, with or without understanding, is the most important goal of their math learning.

Do these conclusions mean that I think

teachers shouldn't teach algorithms? Do they mean that children shouldn't have to learn their basic facts? Do they mean that children shouldn't have to learn to compute? Yes, no, and no.

ALTERNATIVES TO ALGORITHMS

The advantage of algorithms is that they provide reliable ways to compute and, therefore, to simplify potentially difficult calculations. However, no one particular algorithm is best or most efficient; situations and contexts often determine our choice of procedures to use. There is no need for all students to do arithmetic calculations in the same way, any more than it is necessary for all children to develop identical handwriting. A better approach than fitting all children into one arithmetic mold is to make arithmetic an integral part of a problem-solving math curriculum and have children invent their own methods for calculating.

From time to time, some math educators have argued for having children invent their own algorithms. Rob Madell was the mathematics specialist at the Village Community School in New York City for 10 years. In "Children's Natural Processes," which appeared in the March 1985 issue of *Arithmetic Teacher*, Madell argued that "children not only can but should create their own computational algorithms" and that "the teacher's role is 'merely' to help." He described his school's approach of having children figure out their own methods for computing as part of their problem-solving approach to mathematics.

Madell also reported that one striking difference in computational strategies between the children at his school and those who were taught by traditional methods was that children in his program "*universally* proceed from left to right." About the issue of basic facts, Madell wrote: "Eventually, of course, all the facts must be learned. But the early focus on memorization in the teaching of arithmetic thoroughly distorts in children's minds the fact that mathematics is primarily reasoning. This damage is often difficult, if not impossible, to undo."

More recently, Constance Kamii reported some of her research with young children in "Achievement Tests in Primary Mathematics: Perpetuating Lower-Order Thinking," which appeared in the May 1991 issue of *Arithmetic Teacher*. Kamii and her co-author, Barbara Lewis, reported data gathered from comparing second-graders in two schools. One of the schools offered a constructivist primary mathematics program in which teachers had children invent their own procedures for solving computation and story problems. The other school provided traditional instruction in which children were taught algorithms and given opportunities to apply them in exercises and story problems.

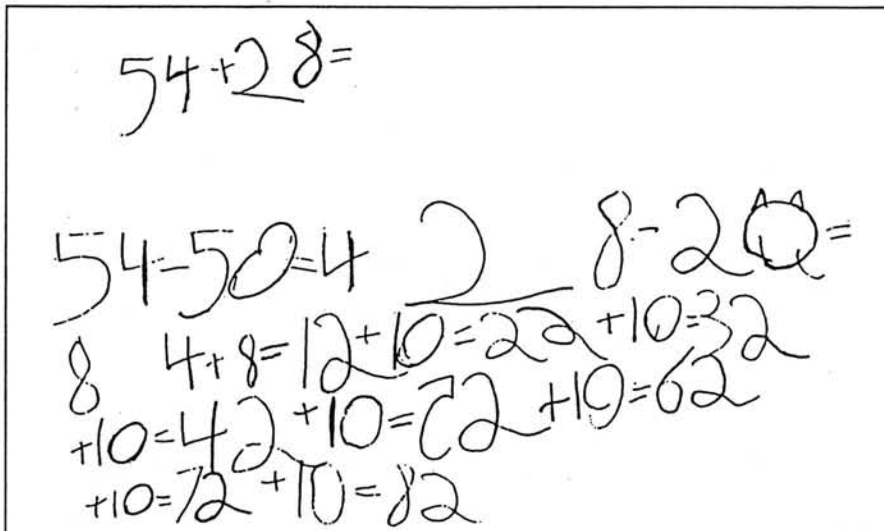
Kamii and Lewis compared achievement test results and also collected interview data on 87 children in four second-grade classes. They analyzed the children's understanding of place value, double-column addition, story problems, mental arithmetic, and estimation. The achievement test scores from both schools were similar, with the children who received traditional instruction scoring slightly higher. For example, the achievement-test cluster called "problem solving" had children solve routine word problems. Out of a possible raw score of 15, the two groups' scores were almost identical: 12.62 (constructivist group) and 12.76 (traditional group).

When the children were asked to explain their thinking, to solve nonroutine problems, and to calculate mentally – tasks calling for higher-order thinking – the children who had not been taught algorithms did significantly better. When asked to mentally calculate 98 + 43 and 3 x 31, 17% of the children from the traditional classes answered each item correctly, while 48% of the children from the constructivist classes answered the first correctly, and 60% answered the second correctly. When asked to figure out how many cars were needed to take 49 children to the zoo when five children could fit in each car, 29% of the children in traditional classes answered correctly, while 61% of the children in constructivist classes answered correctly.

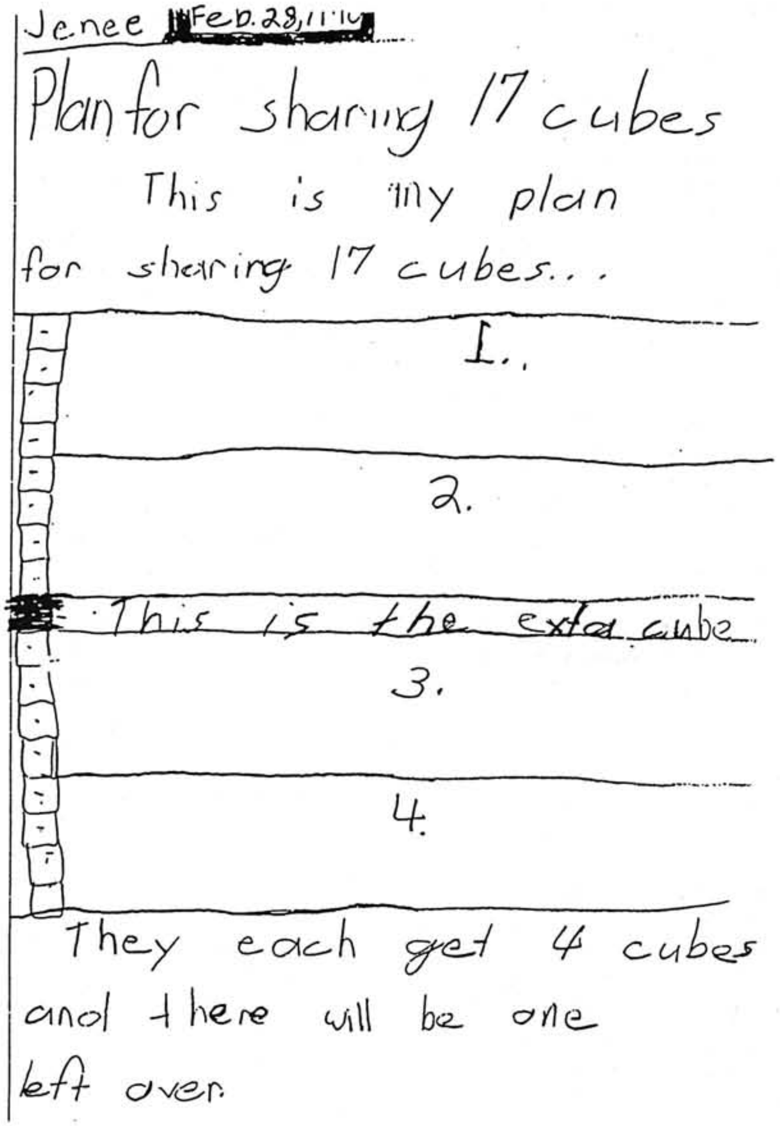
The implication of this research for classroom teaching is that the emphasis in arithmetic instruction should be on having students invent their own ways to compute, rather than learn and practice procedures imposed by the teacher or textbook. Children should not only be challenged to figure out their own methods for calculating, but also be required to explain the reasoning behind their invented procedures. In this alternative approach, time must be provided for students to present their methods to the class. Describing their methods helps students solidify their thinking, while also giving them the opportunity to learn from one another.

A SECOND-GRADE CLASS

Teaching mathematics to second-graders for the past two years has provided me firsthand experience to explore the findings of Rob Madell and Constance Kamii. I do not teach the children any al-



The emphasis in arithmetic instruction should be on having students invent their own ways to compute.



The students were asked to figure out how to share 17 cubes among four children. Here is Jenee's method.

gorithms. Instead, I give them many opportunities to figure out their own ways to deal with numerical calculations in problem-solving situations. For example, one day the children had to figure out the total number of buttons on all their clothing. On other days, I gave them similar problems of investigating the number of pockets, fingers, toes, letters in their first names, books in the class library, and so on. In one lesson, I brought an empty fishbowl to class and had the students each put three cubes in it. I then gave them the problem of figuring out how many "fish" there were in the bowl altogether. In subsequent lessons, they each put in four, five, and 10 fish and solved the problem again.

There were many other opportunities for the class to think about numbers. The children figured out the number of wheels there were on six bicycles and three tricycles, the number of feet and tails there were on five cows and four chickens, and the number of animals and people that were possible if there were a total of 10 feet. The children used manipulative materials, drew pictures, and described their reasoning with both words and numbers.

I filled jars of various sizes with cubes, and the children compared the number of cubes in them. I had them draw stars for one minute and count the number they had drawn; then I had them predict how many stars they could draw in two minutes and test their predictions. The children were engaged with these and similar problems throughout the year.

Every spring the second-graders take a typical standardized test. Last year, I wondered what the children would do on the items that consisted of isolated arithmetic problems. They had not been assigned any worksheets during the year. All their number work had been done in the contexts of solving problems or playing games. One day I gave the children an isolated arithmetic problem: 54 + 28. I gave them blank pieces of paper and asked that they explain how they would find the sum. When I analyzed their papers, I discovered eight different methods among the 25 children.

Not all of my students did the addition by proceeding from left to right, as Rob Madell reported his students did. But 14 of the 25 children did so. For example, Kenny recorded: 54 + 28 = 50 + 20 = 70 + 8 = 78 + 4 = 82.

Beau's method was similar, but he described his thinking in words. He wrote: "I took 54 and crost out the 4. That makes 50. I did the same to 28. The 5 10's and anonother 2 makes 70 put in the 8 and add the 4 that maks 82." Although their explanations and presentations differed, seven children used this method.

Gregory's method had an additional twist. He wrote: "50 add 20 = 70 add 4 + 8 = 12 10 from the 12 = 2 Put the 10 on the 70 = 80 + 2 = 82."

Marissa started with 54. She explained: "I took 54 and add 20 and got 74 and added 8 more and got 82."

Nat used a different approach. He wrote: "54 − 50 = 4 28 − 20 = 8 4 + 8 = 12 + 10 = 22 + 10 = 32 + 10 = 42 + 10 = 52 + 10 = 62 + 10 = 72 + 10 = 82."

Four children did the calculation by counting. Amanda wrote: "I just counted from 54 to 82 thats how I got to 82."

Dustin and Kelly used the standard algorithm. Dustin explained that his uncle had taught him how to do it. Kelly sat next to Dustin.

Five children got incorrect answers or submitted incomplete work. This didn't surprise me. Even when teachers teach the standard algorithm, some children will make errors or have difficulty learning. But when teachers teach the standard algorithm, it's unlikely that a class set of papers would produce the variety of methods shown in the approaches of these children. The class did fine on the standardized test.

THIRD-GRADERS LEARN DIVISION

Dee Uyeda's third-graders in Mill Valley, California, learned about division through a problem-solving approach. Working in pairs and small groups, the children were asked to find solutions to division problems and explain their reasoning. Some problems had remainders, and some didn't. For example, the students were asked to figure out how to share 17 cubes among four children, describe their method, and then use the cubes to test their solution.

Several of the children added to get the answer. For example, Verity wrote: "Each kid gets 4 cubes and 1 goes to the classroom. I figured it out by 4 plus 4 is 8 and 8 plus 8 = 16."

Elliot wrote: "I'm going to do it in 2's. I will count 8 2's and have 1 left. Each child would get 4 cubes."

Joel used what he knew about multiplication. He wrote: "First I just gave one of the cubes to the good will. Then I divided 16 cubes. Because I know that 4 x 4 = 16 so 4 kids each get 4 cubes."

Jenee drew 17 cubes in a vertical line and then counted to divide them into four parts. She wrote: "They each get 4 cubes and there will be one left over."

Rebecca wrote: "I'm going to draw pictures of 17 cubes and four baskets. I put one cube at a time in the basket. Each basket gets four cubes if you want it to be even. Since you have sixteen cubes in all the baskets, the extra cube can go to someone."

Lisa decided to use a calculator. Her writing explains how she coped with her limited understanding of decimals. She wrote: "We each get 4. And put one in the box. How I did it: On the calculator, I pressed 17 − 4 = that didn't work. Then I pressed 17 ÷ 2 = 8.5. .5 is a half. Two halfs is a whole. That is 8 + 9. Half of 8 is 4. And there is one left over."

Some of the division problems presented to the children involved money. For example, in one lesson the children were asked to share $5 among four children. This problem was solved correctly by all groups, and Michelle, Michael, Timothy, and Alana's solution was typical. They wrote: "Each person gets $1.25. We think this because if each person got a dollar there would be one dollar left. And there are four quarters in a dollar. So everybody gets a $1.25."

As the groups finished their work, they were given the problem of sharing $.50 among four children. Children found this problem more challenging. The numbers were more difficult for them, and they had to decide what to do with the remainder. For their solution, the same group wrote: "We think each person gets 12¢ and there would be 2¢ left over that they could not split up, but they could buy bubble gum with the two cents and split the gum. We think this because we have to share the last two cents."

Whenever possible, classroom situations can be used to pose problems. For example, the children counted and found that there were 163 pencils in the class supply. They were given the problem of figuring out how many pencils each child would receive if the pencils were divided equally among them. Laura, Teddy, and Grace drew 27 circles, one for each student, and used tally marks to distribute the 163 pencils. They wrote: "Everybody would get 6 and there would be 1 left over. We figured this out by drawing 27 circles. Grace put tally marks in them while Teddy and Laura counted. We proved it by adding 27 6 times and adding one."

Kendra, Bryce, and Marina wrote: "We think that each child will get 6 pencils and there will be 1 left over. We think this because we made a circle for each kid and gave them each five pencils. We added it up. It came to 135 so we took 135 from 163 and there were 28 left. There are 27 kids in the class so each kid gets one more and there is one left."

After children solved the problems, they presented their results and methods to the class and discussed the different methods used. After discussing each problem, the children were shown the standard notation for representing division problems. With more experience, the children began to use the standard symbols in their own writing.

PERCENTAGES IN MIDDLE SCHOOL

When Cathy Humphreys taught percents to her seventh- and eighth-graders in San Jose, she did not teach them algorithms for solving the three standard types of percentage problems. As described in *A Collection of Math Lessons from Grades 6 Through 8*, by Marilyn Burns and Cathy Humphreys (Cuisenaire Company of America, Inc., 1990), Cathy organized the unit on percentages around a series of problem-solving situations that called for applying percentages. Students worked in small groups and presented their answers and methods to the class. For all problems, Cathy kept the emphasis on making sense of the situation and justifying the methods they used for their calculations.

One such problem, created by Lynne Alper at the EQUALS Project, housed at the Lawrence Hall of Science in Berkeley, California, was the following: "A school has 500 students. If a school bus holds 75 students, is there enough room on one bus for all of the school's left-handed students?"

Cathy gave the class the information that from 12% to 12½% of Americans are left-handed. After collecting information to compare their class data with the national statistic, students worked in pairs to solve the problem. They produced a variety of solutions.

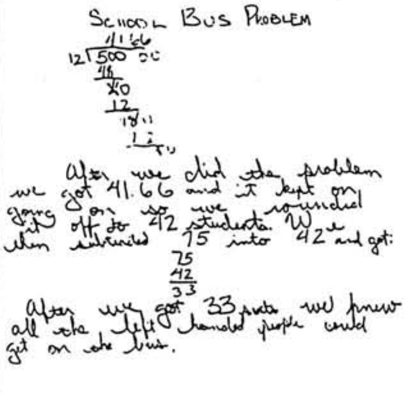
Martin and Tony wrote: "There will be 60 left-handed students on the bus. Out of 100 12% would be 12 people. Since 500 is 5 x more than 100 you times 12 x 5 = 60."

Marshal and Kiet wrote: "Yes, there are enough seats to hold all of the left-handed people because 10% of 500 is 50 people, 2% of 500 is 10 people, so 50 plus 10 is 60 people, and a bus holds 75 people."

Liz and Audrey wrote: "To get the answer we multiplied 500 students by 12% and got 60 people and the bus can hold 75 people so there is enough room."

Khalil and Gina took a completely different approach. They wrote: "We think you can because 75 is 15% of 500. We only have to put 12% on of the left-handed people."

Not all students' methods were appropriate. Jon and Phi, for example, divided 500 by 12 and wrote: "After we did the problem we got 41.66 and it kept on going on so we rounded it off to 42 students. We then subtracted 75 into 42 and got 33. After we got 33 seats we knew all the left handed people could get on the bus."



Not all students' methods were appropriate.

Raymond, Paula, and Stephanie also used division, but they divided 12 by 500. They wrote: "Yes, 12 ÷ 500 = 0.024 so out of 500 students 24 of them are left handed so the bus can hold all the left handed people." Even though their reasoning was erroneous, the students who used division arrived at the correct conclusion. Correct answers can hide a lack of understanding, which is one reason for being sure to have students explain their thinking.

This lesson occurred near the beginning of the unit, and Cathy expected this sort of confusion. As stated in the *Mathematics Model Curriculum Guide*, published by the California State Department of Education, "We must recognize that partially grasped ideas and periods of confusion are a natural part of the process of developing understanding." Cathy led a class discussion during which students presented their methods and the class discussed them. She kept the focus of the discussion on making sense of the procedures presented.

From many experiences with problems such as this one, students began to formulate their own understanding of how to work with percentages. Teachers often fear that, if they don't teach the standard algorithms for percentage problems, students won't learn to solve them. However, the reverse may be a greater worry. Teaching the standard procedures for percentage problems can result in students' not being prepared to *reason* with percentages to solve problems.

A BRITISH mathematician, W. W. Sawyer, pointed out the risk of traditional instruction. "The depressing thing about arithmetic badly taught," he wrote, "is that it destroys a child's intellect and, to some extent, his integrity. Before they are taught arithmetic, children will not give their assent to utter nonsense; *afterwards* they will. Instead of looking at things and thinking about them, they will make wild guesses in the hopes of pleasing a teacher."

The change from teaching standard algorithms to having children invent their own methods requires a major shift for many teachers. It requires first that teachers value and trust children's inventiveness and ability to make sense of numerical situations, rather than their diligence in following procedures. It requires a total commitment to making thinking and reasoning the cornerstone of mathematics instruction. It also requires teachers to be curious about children's ideas, to take delight in their thinking, and to encourage their creativity.

Teachers need support and preparation if they are to adopt these changes. Instructional materials must provide classroom-tested lessons that are consistent with the new guidelines. New tests must change the focus of assessment from mastery of skills to demonstration of understanding. Staff development must be available to provide professional settings for teachers to learn methods for implementing the new instructional changes.

The challenge of teaching math today is to help students gain both confidence and competence in doing mathematics. As stated in the NCTM's *Curriculum and Evaluation Standards for School Mathematics*, doing mathematics requires that children "explore, justify, represent, solve, construct, discuss, use, investigate, describe, develop, and predict." This recommendation demands that we rethink arithmetic instruction. K